Exercises – Special Relativity Revision

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1 Pre-Relativity

1. Consider a transformation of coordinates obtained by a constant translation with a Galileo transformation (i.e. the most general transformation leaving Newton's equations invariant without rotations). Show that the composition of two such transformations is a transformation of the same type. Identify the translation and velocity of the composed transformation.

2. (Lecture exercise) Consider two particles with position vectors $\vec{r}_p$ and $\vec{r}_q$ interacting through a general force of the type
$$\vec{F}_p = -\vec{F}_q(|\vec{r}_p - \vec{r}_q|, |\vec{v}_p - \vec{v}_q|)$$
Show that applying a general transformation as in Ex.1, or a rotation to each particle, then
$$R.\vec{F}_{p/q}(|\vec{r}'_p - \vec{r}'_q|, |\vec{v}'_p - \vec{v}'_q|) \equiv \vec{F}'_{p/q} = m_{p/q}\frac{d^2\vec{r}'}{dt'^2}$$

3. Consider the group of rotations in 3D:
   (a) Show that the product of two rotation matrices is still a rotation matrix, using the two defining conditions (hint: the second condition excludes reflections).
   (b) Consider the following vectors and matrices in index notation:
   $$V^i \rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad W_i \rightarrow (0, 0, -5) \quad M_{ij} \rightarrow \begin{pmatrix} -2 & 7 & 2 \\ 0 & 0 & 3 \\ 4 & -5 & 0 \end{pmatrix}$$
   Compute $V^i W_i$ (what can you say about the result), $V^i M_{ij}$and $M_{ij} V^i$.
   (c) Repeat the proof of the first condition in part (a) using index notation.
   (d) Write a rotation matrix corresponding to a rotation around the z axis and identify the rotation angle.
   (e) Compute the composition of two rotation matrices of the type in (d) and show explicitly what the angle corresponding to the composed rotation is.

2 Lorentz transformations & Minkowski space

1. Consider the group of Lorentz transformations.

   (a) Show, first in matrix notation, that the composition of two Lorentz transformations is still a Lorentz transformation. Repeat the proof using index notation.

   (b) (Index notation) Given the following:

$$A^\alpha \to \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \\ -9 \end{pmatrix} \qquad B_\alpha \to (5, -2, -1, -7) \quad C_{\alpha\beta} \begin{pmatrix} 1 & 3 & 6 & -1 \\ 0.1 & 8 & 4 & 2 \\ 0 & 0 & 0.3 & -3 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

   Compute A_α, B^α, $C^\alpha_{\ \beta}$, $C_\alpha^{\ \beta}$, $C^{\alpha\beta}$, $A^\alpha B_\alpha$ and $C_{\alpha\beta}A^\beta$.

   (c) (Lecture Exercise) Using the definitin of Lorentz transformations $\mathbf{\Lambda^T}\eta\mathbf{\Lambda} = \eta$, show that the most general transformation $\mathbf{\Lambda}$ involving t, x (i.e. y, z not transformed) is

$$\mathbf{\Lambda} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \phi \in \mathbb{R}$$

   which implies

$$\begin{aligned} x^{0'} &= ct' = \cosh\phi\, ct - \sinh\phi\, x \\ x^{1'} &= x' = -\sinh\phi\, ct + \cosh\phi\, x \\ x^{2'} &= y' = y \\ x^{3'} &= z' = z \end{aligned}$$

   Also show that the velocity of the origin of the moving frame O' defined by being at rest in O' (i.e. $x' = 0$) moves with velocity $v \equiv c\tanh\phi = \tanh\phi$ $(c = 1)$, so that we can write

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \gamma \equiv \frac{1}{\sqrt{1 - v^2}}$$

   (d) Using the parametrization of a Lorentz transformation on the plane xt (called Lorentz boost), write the transformation with a pseudo-angle ϕ.

   (e) (Lecture exercise) Write the composition of two transformations in (c) for two different pseudo-angles and identify:

      i. The pseudo-angle of the composed transformation.

      ii. The velocity of each transformation and the velocity of the composed transformation in terms of the first two.

      iii. Compare the result with that composition of two Galileo transformations and comment.

      iv. Recover the Galileo transformation by taking the small velocity limit.

(f) Write a Lorentz transformation corresponding to a rotation around the$z$ axis.

  i. Compute the composition of such rotation with a Lorentz boost on the xt plane.
  ii. Compute the composition in reversed order.
  iii. Comment the results of (i) and (ii).

(g) Consider a Lorentz transformation on x, t:

  i. Compute the inverse transformation.
  ii. Show that the inverse of (i) is still a Lorentz transformation.
  iii. Consider a general Lorentz matrix and its (formally defined) inverse. Show that the inverse is still a Lorentz transformation (i.e. it still obeys the orthogonality relation with respect to $\eta_{\mu\nu}$ and the determinant condition).

2. Show that a the sum of two future directed timelike vectors is still a future directed timelike vector.

3. Consider a space-time diagram $x^0 x^1 (t{-}x\ )$ in an inertial frame $\mathcal{O}$. Represent in such diagram:

  (a) The trajectory of an observer at rest at $x^1 = 1m$
  (b) The trajectory of a particle moving with a velocity $v = 0.1c = 0.1$, such that when $x^0 = ct = 0m$, the particle goes through $x^1 = 5m$.
  (c) The axis $x^{0'} x^{1'}$ of an observer $\mathcal{O}'$ moving with speed $v = 0.5$ along the x^1 axis relative to $\mathcal{O}$ and with the same origin.
  (d) Consider a generic vector r^μ in such diagram, connecting the origin to an event P:
    i. Represent the points in the diagram corresponding to events separated by a proper time to the origin $r^\mu r_\mu = -(1m)^2$
    ii. Represent the points in the diagram corresponding to events separated by a proper space-like distance to the origin $r^\mu r_\mu = (1m)^2$
  (e) the location of events which are simultaneous for $\mathcal{O}$ at the time $x^0 = 2m$
  (f) the location of events which are simultaneous for$\mathcal{O}'$ when $x^{0'} = 2m$
  (g) the trajectory of a photon which is emitted at $x^0 = -1m$ and $x^1 = 0m$, travels in the negative x^1 direction, is then reflected back in a mirror when arrives as $x^1 = -1m$ so it then travels in the positive x^1direction and gets absorbed at $x^1 = 0.75m$.

4. (Lecture Exercise) Time dilation: Consider 2 events $P^\mu \to (0,0,0,0)$ and $Q^\mu \to (T,0,0,0)$. Represent them in a spacetime diagram, and show that their time separation in a moving frame O' is

$$T' = \gamma T \equiv \frac{1}{\sqrt{1-v^2}} T$$

5. (Lecture Exercise) Space contraction: Consider a rod of length d at rest with endpoints following world lines $P^\mu \to (t,0,0,0)$ and $Q^\mu \to (t,d,0,0)$. Represent the world lines of the endpoints in a spacetime diagram, and show that the length of the rod seen in a moving frame O' is $d' = \sqrt{1-v^2} d$.