A first course on Quantum Field theory
Special relativity revision

Marco Sampaio – msampaio@ua.pt

16 de Outubro de 2012

1 Pre-relativity

By the end of the 19th century, Physics was based on two fundamental pillars which were thought to describe all physical phenomena known to date. One of such pillars was the set of Newton's laws which described all mechanical phenomena from the motion of planets (at the scale of the solar system), projectile motion, fluid dynamics etc... On another hand, Maxwell's electromagnetism was able to explain all electric and magnetic phenomena, as well as to provide the right hand side of the Newton equations in certain systems, and also explaining light as a wave phenomenon.

Special relativity arises as a conflict between both theories, at the theoretical as well as experimental level! The way in which this happens is related to the group of transformations which leave each theory invariant. We will see next, that Newton's laws are invariant under a group of coordinate transformations (the Galilean group) when written in inertial reference frames. On the other hand, since the electromagnetic wave phenomenon of light is described by a wave equation, it was natural at the time, to think about it in analogy to fluid dynamics (which transforms under the Galilean group since it comes from Newton's equations effectively), and to assume that there was a fluid on which it propagated dubbed "ether". This raised the question of whether the speed of light should be transformed between frames according to the Galilean laws, just like for fluid dynamics.

1.1 Newtonian mechanics and inertial frames

Newton's laws, express a relation between the set of forces applied on a system of particles, and the trajectories that the particles will follow. In its simplest form (here we label a given particle by p). Newton's second law applied to a particle in the system is

$$\vec{F}_p = m\frac{d^2\vec{r}_p}{dt^2} = m\frac{d\vec{v}_p}{dt} = m\vec{a}_p \tag{1}$$

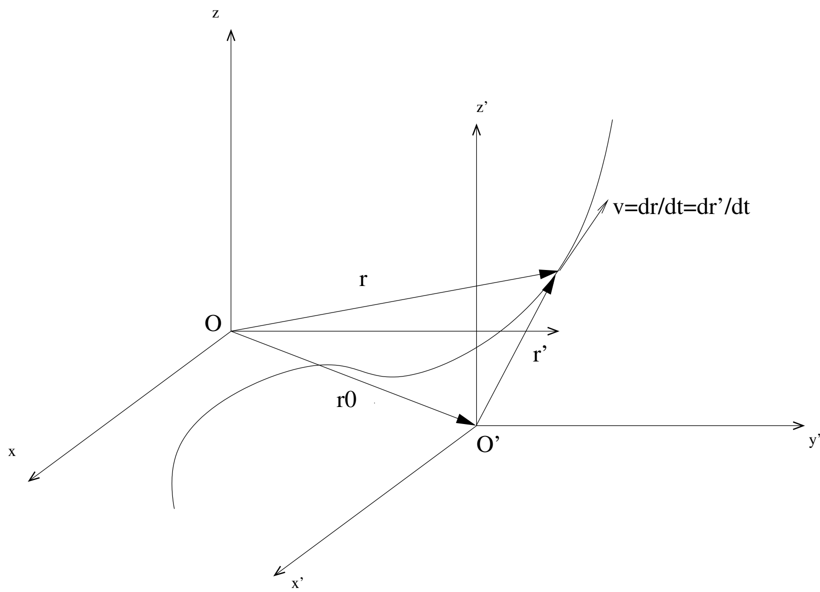


Figure 1: Constant translations of the coordinate system.

where the vector of forces applied on the particle is $\vec{F_p} = (F^x, F^y, F^z)$, the vector $\vec{r_p} = (x, y, z)$ is the position of the particle with respect to the origin of the coordinate system O, and we have also rewritten the right hand side in terms of the velocity and acceleration. This law defines the inertial frames. For example, in an inertial frame, Newton's first law of inertia says that the total force acting on a object in uniform motion with constant velocity is zero. This is readily obtained from the Newton law Eq.(1), so the inertial frames are those in which the law of inertia is obeyed by the equations of motion.

1.1.1 Invariance under translations and rotations

The group of invariance of Newton's laws is directly related with the fact that the right hand side of the equations is differential in time. Thus, by looking to the right hand side it is easy to see that a transformation of the form

$$t' = t + t_0 \tag{2}$$

$$\vec{r'_p} = \vec{r_p} + \vec{r_0} \tag{3}$$

doesn't affect the equations of motion. In fact, if one calculates the right hand side of Newton's equations in reference frame O'

$$\frac{d\vec{r_p}'}{dt'} = \frac{dt}{dt'}\frac{d(\vec{r_0} + \vec{r_p})}{dt} = \frac{d\vec{r_p}}{dt} \Rightarrow \frac{d^2\vec{r_p}'}{dt'^2} = \frac{d^2\vec{r_p}}{dt^2} \tag{4}$$

thus the right hand side of Newton's equations is invariant under constant translations. Concerning the left hand side (the applied forces) in general one can have, for each

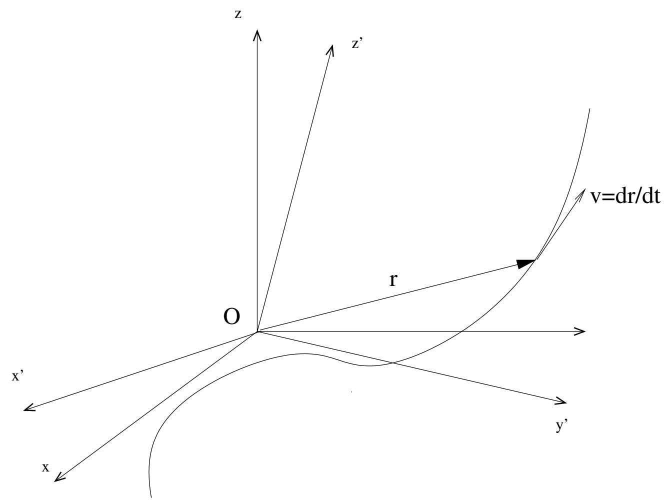


Figure 2: Constant rotation of the coordinate system

particle, a force which depends on positions, velocities and time. However, in nature, for closed systems of particles, that is without external forces being applied, in general, the force applied to a given particle depends only on the difference between positions and velocities of pairs of particles ($\vec{r}_q$ and $\vec{v}_q$ represent the position and velocity of some other particle in the system)

$$\vec{F}_p(|\vec{r}_p - \vec{r}_q|, |\vec{v}_p - \vec{v}_q|) \tag{5}$$

Since a translation of the coordinate system, affects all position vectors in the same way, the difference is clearly invariant. Physically, this property is expected, since the description of our physical system should not depend on the point in space that we are taking as origin, or the time when we decide that the clocks start ticking. A simple example of such a force is Newton's law of gravitation, in which the force between test masses only depends on the distance between them, and the same for the electrostatic force between electric charges.

Very similarly, one can show that a constant rotation of the coordinate system, does not affect Newton's equations. That is also expected since that orientation of the axes should also be arbitrary. The proof follows in a similar way, considering that a rotation of a vector is given in general by a rotation matrix $\mathbf{R}$ acting on a column vector

$$\vec{r}'_p = \mathbf{R}.\vec{r}_p \tag{6}$$

More explicitly

$$\begin{pmatrix} x'_p \\ y'_p \\ z'_p \end{pmatrix} = \begin{pmatrix} R_{x'x} & R_{x'y} & R_{x'z} \\ R_{y'x} & R_{y'y} & R_{y'z} \\ R_{z'x} & R_{z'y} & R_{z'z} \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} \tag{7}$$

To show invariance, one has to determine the acceleration in reference frame $\mathcal{O}$, as a function of the acceleration in frame $\mathcal{O}'$, insert in Newton's equations of motion and show that the equations take the same form when expressed in terms of the accelerations and force in frame $\mathcal{O}'$. If the matrix is constant,

$$\frac{d^2\vec{r}'_p}{dt^2} = \mathbf{R}.\frac{d^2\vec{r}_p}{dt^2} \Rightarrow \frac{d^2\vec{r}_p}{dt^2} = \mathbf{R}^{-1}.\frac{d^2\vec{r}'_p}{dt^2} \tag{8}$$

inserting in Newton's equations

$$\begin{aligned} \vec{F}_p &= m\mathbf{R}^{-1}.\frac{d^2\vec{r}'_p}{dt^2} && (9)\\ \Leftrightarrow \mathbf{R}.\vec{F}_p &= m\frac{d^2\vec{r}'_p}{dt^2} && (10)\\ \Leftrightarrow \vec{F}'_p &= m\frac{d^2\vec{r}'_p}{dt^2} && (11) \end{aligned}$$

where we defined the rotated force $\vec{F}'_p = \mathbf{R}.\vec{F}_p(|\vec{r}_p - \vec{r}_q|, |\vec{v}_p - \vec{v}_q|) = \mathbf{R}.\vec{F}_p(|\vec{r'}_p - \vec{r'}_q|, |\vec{v}'_p - \vec{v}'_q|)$, and in this last step we have used the fact that the norm of vectors is invariant under rotations.

Note that, however, if the rotation is time dependent, in general Newton's equations will take a different form, and the new frame will not be inertial (new forces such as the Coriolis and centrifugal force will appear).

1.1.2 Invariance under Galilean transformations

Since the right hand side of Newton's equations contain second order time derivatives of the position vector, one also has the freedom to make a constant translation of the velocity, that is, to consider a moving frame with uniform velocity. Such a transformation is called a Galileo transformation, which takes the form

$$\vec{r}'_p = \vec{r}_p + \vec{v}_0 t \tag{12}$$

thus, for example, if one considers $\vec{v}_0 = (-v_x, 0, 0)$ then, the moving frame is moving towards the right with constant velocity. Clearly the right hand side of Newton's equations is invariant. As for the forces, again, for a closed system the $\vec{v}_0 t$ cancels in the difference between vectors.

1.1.3 Summary of the invariance of Newton's laws and composition of transformations

In summary, the group of transformations that leave classical mechanics invariant are split in two types:

1. Up to linear in time and space translations of the form

$$\begin{aligned} t' &= t_0 + t && (13)\\ \vec{r}' &= \vec{r} + \vec{r}_0 + \vec{v}_0 t \Leftrightarrow \begin{array}{c} x' = x + x_0 + v_0^x t \\ y' = y + y_0 + v_0^y t \\ z' = z + z_0 + v_0^z t \end{array} && (14) \end{aligned}$$

2. Constant rotations

$$\vec{r}' = \mathbf{R}.\vec{r} \tag{15}$$

In particular, the composition of two such transformations is also a transformation of the same type.

1.1.4 The group of rotations

A convenient (and in fact rigorous) way of thinking about the rotation group, is in terms of scalar products between vectors. Two essential properties of rotations are:

- They leave the length of vectors invariant, that is, if we apply a rotation to a vector, its norm stays the same
- The angle between vectors also stays invariant

Both properties are controlled by the scalar product between vectors. So it is natural to define rotations as those linear transformations which leave the scalar product between vectors invariant. The scalar product between vectors $\vec{r}$ e $\vec{w}$ can be represented in matrix notation as the product between a row and a column vector

$$\vec{r} \cdot \vec{w} \to r^T w = (r_x, r_y, r_z) \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

If one defines a rotation matrix $\mathbf{R}$ acting on a vector w in frame $\mathcal{O}$, returning a vector w' in the rotated frame $\mathcal{O}'$, then

$$w' = \mathbf{R}w \; .$$

Invariance of the scalar product means that the scalar product between any two vectors in frame $\mathcal{O}$ is equal to the scalar product between the rotated vectors in frame $\mathcal{O}'$, i.e.

$$r'^T w' = r^T w \; .$$

(In special relativity we will also define a space-time scalar product). Using the definition of rotations on the left hand side

$$\begin{aligned} (\mathbf{R}r)^T \mathbf{R}w &= r^T w \\ r^T \mathbf{R}^T \mathbf{R}w &= r^T w \end{aligned}$$

Thus, for the scalar product to be invariant for any two vectors we need

$$\mathbf{R}^T\mathbf{R} = \mathbf{1}$$

and the rotation matrix is said orthogonal. Another condition, which is necessary to exclude reflections (which are also orthogonal but invert the axes) is $\det \mathbf{R} = 1$. Note that from the orthogonality relations above one shows

$$(\det \mathbf{R})^2 = 1 \Rightarrow \det \mathbf{R} = \pm 1 \tag{16}$$

The positive sign is associated with rotations and the negative to reflections. For instance, a rotation matrix on the xy plane is given by

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

An example of a reflection matrix which inverts the x axis is

$$\mathbf{Inv} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Last lecture: Motivation

Combines:

$$\boxed{\text{Quantum Field Theory } (c, \hbar)} \begin{array}{l} \rightarrow \boxed{\text{Special Relativity } (c)} \leftarrow \\ \rightarrow \text{Quantum Mechanics } (\hbar) \end{array}$$

↳ Important • High energy particles @ scales smaller than proton.

2 Revision of Special Relativity

Newton Laws (Galilean group of invariance)		Maxwell's electromagnetism & wave equation for light
• Planets • Projectiles • Fluid dynamics	→ Theory/Exp. conflict ← ⇓ **Special relativity**	• Electricity • Magnetism • Light

2.1 Newton's laws, galilean transformations & index notation

$$\boxed{\vec{F} = m\frac{d^2\vec{x}}{dt^2}}$$ & defines inertial frames where law of inertia $\vec{F} = 0 \Rightarrow \vec{v} = \text{const.}$

→ Transform. leaving it invariant (Galilean group)

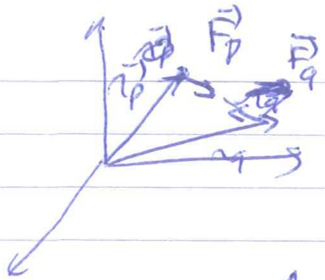
1) Time and space translations up to linear order in t:

$$t' = t_0 + t \quad ; \quad \vec{x}' = \vec{x}_0 + \vec{x} + \vec{v}_0 t$$

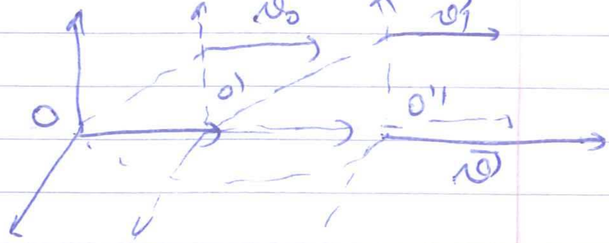
2) Constant rotations.

$$\vec{x}' = R\vec{x} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} R \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Exercise 2



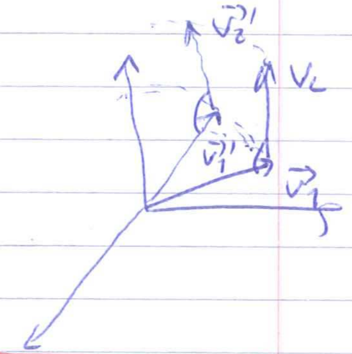
Note on particular:



$$\boxed{\vec{v} = \vec{v}_0 + \vec{v}'}$$ Addition of velocities

2.1.1 A detour through rotations (index notation)

Properties of R:



- Leaves the norm invariant
- Leaves angles invariant

→ Controlled by scalar product ⇒ $\boxed{\text{R leaves scalar product invariant!}}$

$$\vec{v} \to \begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix} \to \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \to v^i \quad i=1,2,3 \qquad w^i \to \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}$$

$$\vec{v} \cdot \vec{w} = |v||w|\cos\theta$$

$$= v^1 w^1 + v^2 w^2 + v^3 w^3$$

$$= \sum_{i=1}^{3} v^i w^i = \sum_{i,j=1}^{3} v^i \delta_{ij} w^j$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix notation:

$$\vec{v} \cdot \vec{w} = (v^1 \; v^2 \; v^3) \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}$$

define row vector $v_j = \sum_{j=1}^{3} v^i \delta_{ij}$

indices up $\Rightarrow$ column vectors
indices down $\Rightarrow$ row vectors (transposed)

$V_j \rightarrow$ numerically the same as V^j for now...

<u>Einstein convention</u>: Repeated indices are summed over
$\Rightarrow$ can ommit $\sum_1^3$ (contraction)

Then $\vec{V} \cdot \vec{W} = V^i \delta_{ij} W^j = V_j W^j$

⚠ Beware ~~$\vec{V}$~~ $\vec{V} \cdot \vec{W} \times \vec{X} \cdot \vec{Y} = V_j W^j \; X_k W^k$
~~(not~~
(not $V_j W^j \underbrace{X_j Y^j}_{\text{ambiguity}}$)

→ Back to rotations:

$$\vec{V}' = R\vec{V} \qquad \begin{pmatrix} V^{1'} \\ V^{2'} \\ V^{3'} \end{pmatrix} = \begin{pmatrix} & R^{i'}{}_{j'} & \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

index $V^{i'} = \sum_{j=1}^{3} R^{i'}{}_j V^j = \boxed{R^{i'}{}_j V^j}$ $\left(\begin{array}{l} \vec{V}' = R\vec{V} \\ \vec{V}'^T = \vec{V}^T R^T \end{array} \right)$

Also $W^{i'} = R^{i'}{}_j V^j$

~~R invariance~~
Scalar product invariance $\Rightarrow \vec{V}' \cdot \vec{W}' = \vec{V} \cdot \vec{W}$

• Matrix notation:

$$(V^{1'} V^{2'} V^{3'}) \begin{pmatrix} W^{1'} \\ W^{2'} \\ W^{3'} \end{pmatrix} = (V^{1'} V^{2'} V^{3'}) \begin{pmatrix} W^{1'} \\ W^{2'} \\ W^{3'} \end{pmatrix}$$

$$\Rightarrow (V^1 V^2 V^3) R^T R \begin{pmatrix} W^1 \\ W^2 \\ W^3 \end{pmatrix} = (V^1 V^2 V^3) \begin{pmatrix} W^1 \\ W^2 \\ W^3 \end{pmatrix}$$

$\Rightarrow \boxed{R^T \mathbb{1} R = \mathbb{1}}$ → orthogonal matrices

Now, index notation:

$$V^{i'} = R^{i'}{}_{l} V^{l} \Leftrightarrow V_{i'} = R_{i'l} V^{l}$$

$$W^{j'} = R^{j'}{}_{k} W^{k}$$

- $\vec{V}' \cdot \vec{W}' = \vec{V} \cdot \vec{W}$

$$\Leftrightarrow V_{i'} W^{i'} = V_i W^i$$

$$\Leftrightarrow R_{i'l} V^{l} R^{j'}{}_{k} W^{k} = V_i W^i$$

$$\Leftrightarrow V^{l} R_{i'l} R^{j'}{}_{k} W^{k} = V_i W^i$$

$$\Leftrightarrow V^{l} R^{T}{}_{li'} R^{i'}{}_{k} W^{k} = V^{j} \delta_{ji} W^{i}$$

$$\Leftrightarrow V^{l} \underbrace{R^{T}{}_{li'} R^{i'}{}_{k}} W^{k} = V^{l} \underbrace{\delta_{lk}} W^{k}$$

$$\Leftrightarrow \boxed{R^{T}{}_{li'} R^{i'}{}_{k} = \delta_{lk}} \qquad (R^T R = \mathbb{1})$$

$$\Downarrow$$

$\det R = \pm 1$ — $+1$: rotations, -1: reflections

2.2 Conflict with electromagnetism & Michelson-Morley experiment

$\boxed{\text{Maxwell equations in vacuum}}$ $\Longrightarrow$ wave equations

$$\left\{ \begin{array}{l} \Delta \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \\ \Delta \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \end{array} \right\} \text{propagation of light}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

Ex: plane waves $\phi / \vec{A} \sim e^{i(\vec{k}\cdot\vec{x} - \omega c t)}$

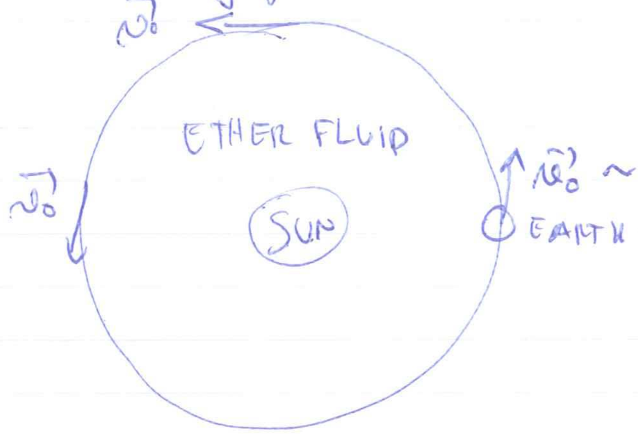
$\Rightarrow$ Analogous to $\underbrace{\text{fluid dynamics}}$ $\vec{v} = \vec{c}$

$\hookrightarrow$ Newtonian $\Rightarrow$ $\vec{c}\,' = \vec{v}_0 + \vec{c}$ in a moving frame W.R.T. fluid

Q?: Does the same happen with light?

A: No ⚠

Michelson-Morely experiment measures $\vec{c}$ in different moving frames



$\vec{c}\,' = \vec{c}$ (!) confirmed by all modern experiments too.

Conclusion: • Speed of light same for all inertial observers at ~~$\vec{v}_0$~~ different $\vec{v}_0$.

• Maxwell eqs. same in all frames.

Q?: How to reconcile this with laws of Mechanics?

2.3 Special Relativity postulates

Postulate 1 Galileo's principle: The laws of physics (Maxwell eqs., Mechanics eqs, particle physics eqs. etc...) are the same in all inertial frames (relative $\vec{v}_0$)

Postulate 2: The speed of light in vacuum is the same in all inertial frames.

Comments: • 2) (!) incompatible with Galilean transformations.
$\Rightarrow$ Laws of Mechanics must be adapted

• Both compatible with Maxwell Eqs (already relativistic!)

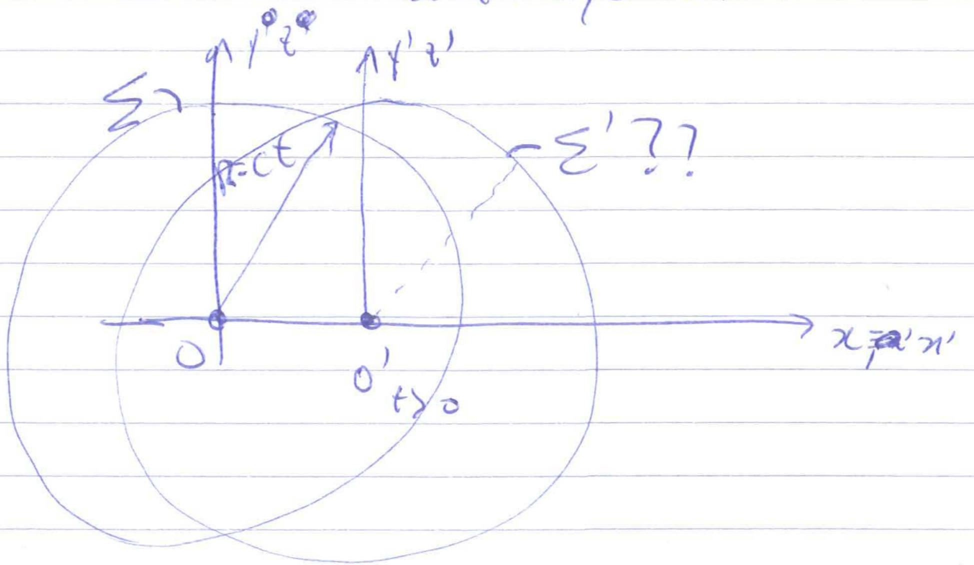
$\Downarrow$

Use light as guide to find new transformation laws (Lorentz) between inertial frames.

12.3.1 ~~Lorentz~~ Lorentz transformations

Consider: • Frame $Oxyz$ (at rest) emits spherical wave at $t=0$

• Moving frame $Ox'y'z'$ with v along x emits at $t=0$, $O \equiv O'$



Σ - wave front $x^2+y^2+z^2=c^2t^2$

Σ' - Same in O' frame! $x'^2+y'^2+z'^2=c^2t^2$

But! if assume Galilean transformation.

$x'=x-vt$ $\Rightarrow$ $\Sigma' : (x-vt)^2+y^2+z^2=c^2t^2$

$y'=y$

$z'=z$ $\neq x'^2+y'^2+z'^2=c^2t^2$

$\Rightarrow$ Conflict with postulate 1 (law of physics, i.e. wave front cannot depend on relative motion)

$\Rightarrow$ Change ~~trans~~ to Lorentz transf. for the waves to coincide in all frames!

2 Special relativity

In the last section, we have seen that the transformation laws of Newtonian relativity between inertial frames are incompatible with Maxwell's theory when applied to electromagnetic waves in vacuum as to describe light propagation. We have also seen in particular, that interferometer experiments which measure the speed of light imply that it must be a constant regardless of the relative motion between the source and observer.

Nevertheless, the transformation laws that preserve Newton's equations, contain an important principle, which is that the laws of mechanics are independent of the relative motion between inertial frames. This principle is called Galileo's principle and is preserved in special relativity. Special relativity, finds a way to reconcile this principle with the experimental (and to some extent theoretical) fact, that the speed of light in vacuum is a constant.

Postulate 1 - Principle of Relativity The laws of Physics (Maxwell's equations, Mechanics equations, particle Physics equations, etc...) are the same in all inertial frames (with a relative velocity).

This postulate is, in particular, compatible with the fact that Maxwell's equations are the same independently of the relative motion between source and observer. However, this somehow clashes with the Galilean transformations since the speed of light does not get transformed between inertial frames as confirmed by the Michelson Morley experiment. This will imply that we need to change the transformation laws between inertial frames, using light as a guide, and the laws of Mechanics as well.

Postulate 2 - Constancy of the speed of light The speed of light in vacuum is the same in all inertial frames.

This is also obviously compatible with Maxwell's equations. Those equations are actually already relativist, after close inspection, so it should not be surprising a posteriori that all clues for special relativity, came from such theory.

2.1 Lorentz transformations

To reconcile the transformation laws between inertial frames, with the postulates in the introduction, we will analyze the propagation of light. Let us consider:

- A reference frame $\mathcal{O}$ at rest, in which a spherical light wave is emitted at time $t = 0$ by a point like source at the origin,
- a uniformly moving source (frame $\mathcal{O}'$) along the x axis, with velocity v, and that such source also emits a spherical light wave $t = 0$, when its origin coincides with the source at rest ($\mathcal{O} = \mathcal{O}'$ at $t = 0$).

According to postulate 2, the wave must propagate at speed c in both frames. The position of wave front Σ in the frame at rest (Fig. 3) for $t > 0$ after being emitted, is

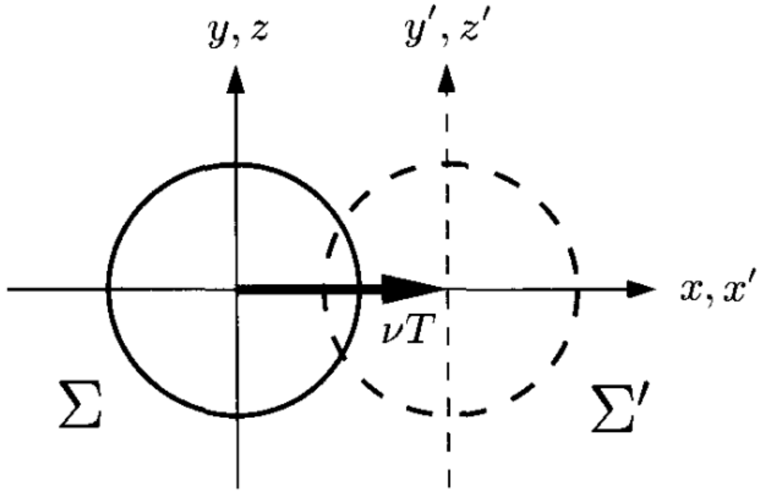


Figure 3: Wave fronts according to a Galilean transformation. (extracted from Stephani, H; Relativity)

given by the equation of a sphere with radius $R = cT$, that is

$$x^2 + y^2 + z^2 = c^2 t^2 \,. \tag{17}$$

The wave front in the moving frame Σ' obeys the same equation is such frame with primed coordinates. Using a Galilean transformation, we would expect

$$\begin{aligned} x' &= x - vt & (18)\\ y' &= y & (19)\\ z' &= z & (20) \end{aligned}$$

and we would obtain instead

$$(x - vT)^2 + y^2 + z^2 = c^2 t^2 \,. \tag{21}$$

Clearly, with this transformation, the second wave front does not coincide with the first one ($\Sigma \neq \Sigma'$). However, this is in conflict with the first postulate, since the two light pulses were emitted simultaneously from the same point, so their propagation (i.e. the associated law of Physics) cannot depend on relative motion. This shows:

- $\Rightarrow$ The need to change the concept of simultaneous events by changing the transformation laws between inertial frames. As a consequence, the laws of mechanics will also be changed as to become relativistic.

Let's rewrite:

Σ: $-c^2t^2 + \underbrace{x^2+y^2+z^2}=0$ ⟵ similar to scalar product δ_{ij}, but

Define $x^{\mu=0,1,2,3} = (x^0, x^1, x^2, x^3)$ → greek indices μ, ν, α, β
$= (ct, x, y, z)$ $\quad$ $0, \underbrace{1,2,3}_{\text{space}}$
time ↗

define $\eta_{\mu\nu}$ → $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Then Σ: $\boxed{x^\mu \eta_{\mu\nu} x^\nu = 0}$ $\left(\Leftrightarrow -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0\right)$

↳ scalar product of vectors in space-time

- Also define lower indices $x_\mu \equiv \eta_{\mu\nu} x^\nu$ ⚠ Now $x_0 = -x^0$!
- Similarly define the inverse $\eta^{\mu\nu} \equiv (\eta^{-1})^{\mu\nu}$ to perform scalar product of lowered indices
(Same entries ↗ as $\eta_{\mu\nu}$ actually)

$\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu{}_\nu$, $\eta^{\mu\nu}$ raises indices $\Rightarrow x^\mu = \underbrace{\eta^{\mu\nu} x_\nu}_{\text{Implicit sum}}$

with this notation:

Σ: $x^\mu \eta_{\mu\nu} x^\nu = 0$
Σ': $x^{\mu'} \eta_{\mu'\nu'} x^{\nu'} = 0$

Now, modify Galileo transformation to be a general linear trans.

$x^{\mu'} = \underbrace{\Lambda^{\mu'}{}_\alpha}_{4\times 4 \text{ matrix}} x^\alpha$ $\quad \begin{cases} x^{0'} = \Lambda^{0'}{}_0 x^0 + \Lambda^{0'}{}_1 x^1 + \dots \\ x^{1'} = \Lambda^{1'}{}_0 x^0 + \Lambda^{1'}{}_1 x^1 + \dots \\ x^{2'} \dots \\ x^{3'} \dots \end{cases}$

then Σ': $\Lambda^{\mu'}{}_{\alpha} x^{\alpha} \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\beta} x^{\beta} = 0$

$\Leftrightarrow$ $x^{\alpha} \Lambda^{\mu'}{}_{\alpha} \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\beta} x^{\beta} = 0$

$\Leftrightarrow$ $x^{\alpha} \underbrace{\Lambda^{T}{}_{\alpha}{}^{\mu'} \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\beta}} x^{\beta} = 0$

But for Σ: $x^{\alpha} \eta_{\alpha\beta} x^{\beta} = 0$

$\Rightarrow$ $\Lambda^{T}{}_{\alpha}{}^{\mu'} \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\beta} = \eta_{\alpha\beta}$

$\boxed{\Lambda^{T} \eta \Lambda = \eta}$ in matrix notation

$\rightarrow$ (compare with $R^{T} \mathbb{1} R = \mathbb{1}$

$\rightarrow$ = group of linear transformations in space-time orthogonal ~~w.r.t.~~. with respect to metric $\eta_{\mu\nu}$
($\eta_{\mu\nu}$ defines scalar product in space-time)

Exercise 1) c) & e) see solutions sheet!